SPRING 2022

SOLUTIONS MATH 290 EXAM 2

No calculators, phones or laptops may be used during this exam.

Name:

(I) **True-False:** Mark each statement below as True or False. (10 points)

(a) $\mathbb{R}^4$ can be spanned by five vectors. True, but one of the vectors will be redundant.

(b) The matrix $\begin{bmatrix} \pi & \sqrt{2} \\ 0 & e \end{bmatrix}$ is diagonalizable. True, since A has distinct eigenvalues π and e.

(c) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^1 & e^2 \\ e^3 & e^4 \end{bmatrix}$. False, as stated many times in class.

(d) If the square matrix B is obtained from A by the row operation $2R_1 + R_2$, then $|B| = 2 \cdot |A|$. False. This row operation preserves the determinant.

(II) **Short answer.** (5 points) Explain why the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.You must justify your answer.

Solution. $c_A(X) = \det \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix} = x^2$, so A has eigenvalue 0 with multiplicity two. On the other hand, E_0 is the null space of the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has a basis consisting of one vector, so that E_0 is one dimensional. Thus, the dimension of E_0 is not equal to the multiplicity of 0 as an eigenvalue, so A is not diagonalizable.

(III) (30 points) Answer each part for the matrix $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 2 & 4 & 6 \end{bmatrix}$: (i) Calculate $c_A(x)$, the characteristic polynomial; (ii) Find the eigenvalues of A (with multiplicity); (iii) Find the bases for each eigenspace of A; (iv) Determine, with justification, if A is diagonalizable; and (v) If A is diagonalizable, find the matrix P (but not P^{-1}) that diagonalizes A. **Be sure to indicate clearly your answer to each part of this question.**

Solution. (i) $c_A(x) = \det \begin{bmatrix} x+2 & 0 & 0 \\ 0 & x+2 & 0 \\ -2 & -4 & x-6 \end{bmatrix} = (x+2)^2(x-6)$, since the matrix is lower triangular.

(ii) The eigenvalues of A are -2 (with multiplicity two) and 6.

(iii) To find a basis for E_{-1}: $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -4 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Basic solutions for the homogeneous system corresponding to this last matrix are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, and these vectors are a basis for E_0.

To find a basis for E_6: $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ -2 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which yields $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as basis for E_6.

(iv) A is diagonalizable since $\dim(E_0) = 2$ which is the multiplicity of 2, and $\dim(E_6) = 1$ which is the multiplicity of 6.

(v) The matrix $P = \begin{bmatrix} -2 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ diagonalizes A.

(IV) (30 points) Consider the system of first order linear differential equations:

$$x_1'(t) = 17x_1(t) - 30x_2(t)$$
$$x_2'(t) = 10x_1(t) - 18x_2(t),$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 1$.

(a) Calculate e^{At}, where A is the coefficient matrix of the system. You may use the facts that the eigenvalues of A are 2, -3, with eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, respectively.

Solution. The diagonalizing matrix is $P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ is its inverse. We also have $e^{Dt} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$. Since $e^{At} = Pe^{Dt}P^{-1}$, we have:

$$e^{At} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^{2t} & 3e^{-3t} \\ e^{2t} & 2e^{-3t} \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4e^{2t} - 3e^{-3t} & -6e^{2t} + 6e^{-3t} \\ 2e^{2t} - 2e^{-3t} & -3e^{2t} + 4e^{-3t} \end{bmatrix}.$$

(b) Use your answer in part (b) to solve the given system with initial conditions, that is find $x_1(t)$ and $x_2(t)$ satisfying the given system and initial conditions.

Solution. Since the solution to the given system is $e^{At} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4e^{2t} - 3e^{-3t} & -6e^{2t} + 6e^{-3t} \\ 2e^{2t} - 2e^{-3t} & -3e^{2t} + 4e^{-3t} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10e^{2t} + 9e^{-3t} \\ -5e^{2t} + 6e^{-3t} \end{bmatrix},$$

and thus, $x_1(t) = -10e^{2t} + 9e^{-3t}$ and $x_2(t) = -5e^{2t} + 6e^{-3t}$.

(V) (25 points) Consider the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Do the vectors v_1, v_2, v_3 form a basis for $\mathbb{R}^3$? You must justify your answer.

Solution. $\det \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 0 + -2 \neq 0$, so the given vectors form a basis for $\mathbb{R}^3$.

(b) If v_1, v_2, v_3 form a basis for $\mathbb{R}^3$, write $w = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$ as a linear combination of v_1, v_2, v_3, otherwise, write one of the vectors v_1, v_2, v_3 as a linear combination of the remaining ones.

Solution. Using Gaussian elimination we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & -4 \end{array}\right] \xrightarrow[R_2+R_1]{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 10 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & -8 \end{array}\right] \xrightarrow{-1 \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 10 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{-2 \cdot R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 8 \end{array}\right].$$

We therefore have $w = -6v_1 - 4v_2 + 8v_3$.

Bonus Problem. (10 points) Suppose A is a 4×4 matrix with $c_A(x) = (x-1)^4$. Give an example of such an A so that:

(i) A is not diagonalizable.
(ii) A is diagonalizable.

Solution. For (i), we can take $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, since 1 is an eigenvalue with multiplicity 4 and the dimension of the eigenspace E_1 equals three. Note that any upper (or lower) triangular matrix matrix with 1s down the diagonal and at least one non-zero entry above the main diagonal will work.

For (ii), we can just take $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, since it is already a diagonal matrix! And this is the only such matrix that will work, since in this case, the dimension of E_1 must be four, which forces $1 \cdot I_4 - A$ to be the zero matrix.